System Dynamics and Mechanical Vibration

If you have a smart project, you can say "I'm an engineer"

Lecture 1

Staff boarder

Dr. Mohamed Saber Sokar

Dr. Mostafa Elsayed Abdelmonem

System Dynamics and Mechanical Vibration

- **Lecture aims:**
  - Formulate the equations of motion of two-degree-of-freedom systems
  - Identify the mass, damping, and stiffness matrices from the equations of motion

Two Degree-of-Freedom Systems

1. Model problem
   - Matrix form of governing equation
   - Special case: Undamped free vibrations
   - Examples
2. Transformation of coordinates
   - Inertially & elastically coupled/uncoupled
   - General approach: Modal equations
   - Example
3. Response to harmonic forces
   - Model equation
   - Special case: Undamped system

Two-Degree-of Freedom Systems

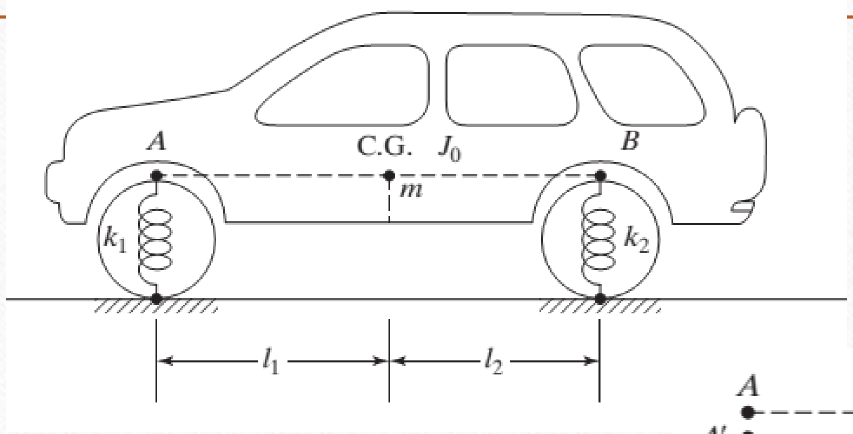
[*k*], and [*c*], [*m*] are called the *stiffness* , *damping*, and *mass matrices* respectively,

$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

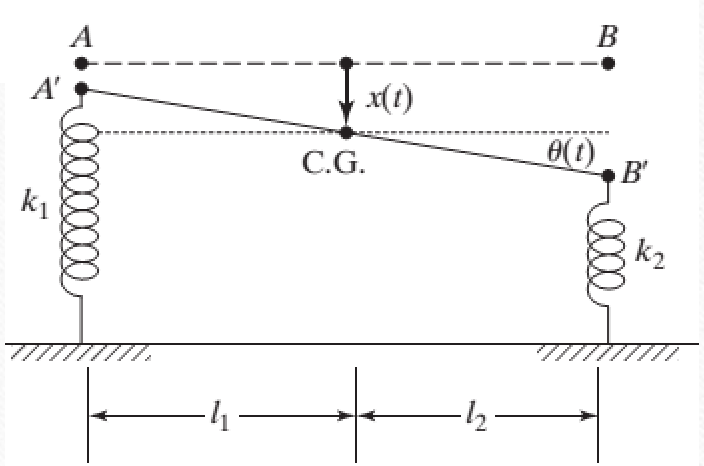
$\vec{x}(t)$ and $\vec{f}(t)$ are called the *displacement* and *force vectors*, respectively.

Two-Degree-of Freedom Systems

Thus the system has one point mass *m* and two degrees of freedom, because the mass has two possible types of motion (translations along the *y* and *x* directions). The general rule for the computation of the number of degrees of freedom can be stated as follows

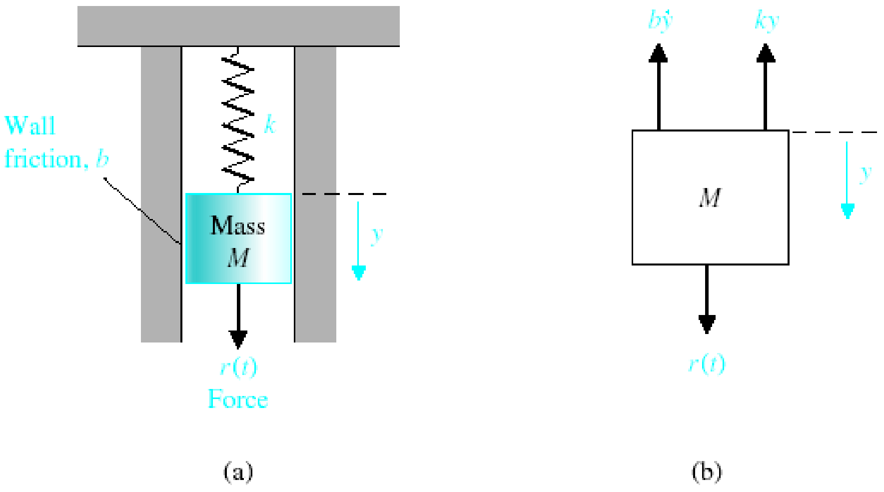


$$\text{Number of degrees of freedom of the system} = \text{Number of masses in the system} \times \text{number of possible types of motion of each mass}$$



Modeling of Mechanical System

- **Spring - Mass - Damper**



(a) Spring-mass-damper system.
(b) Free-body diagram.

$$M \cdot \frac{d^2}{dt^2} y(t) + b \cdot \frac{d}{dt} y(t) + k \cdot y(t) = r(t)$$

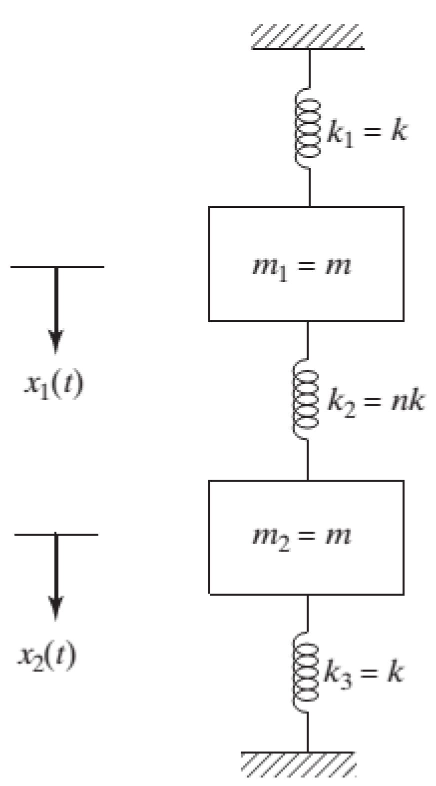
Two-Degree-of Freedom Systems

Equations of motion:

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

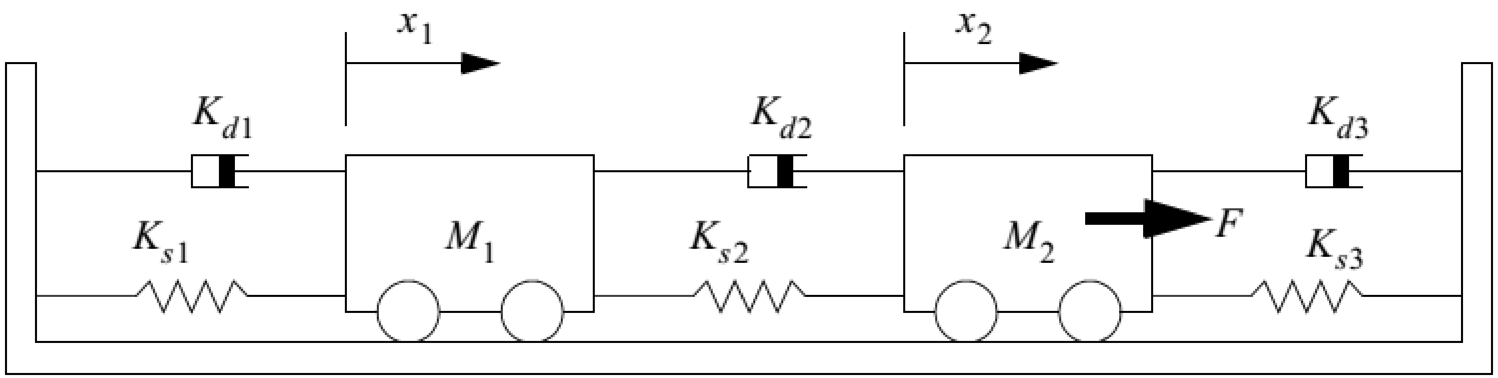
$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

We are interested in knowing whether m_1 and m_2 can oscillate harmonically with the same frequency and phase angle but with different amplitudes. Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle ϕ,

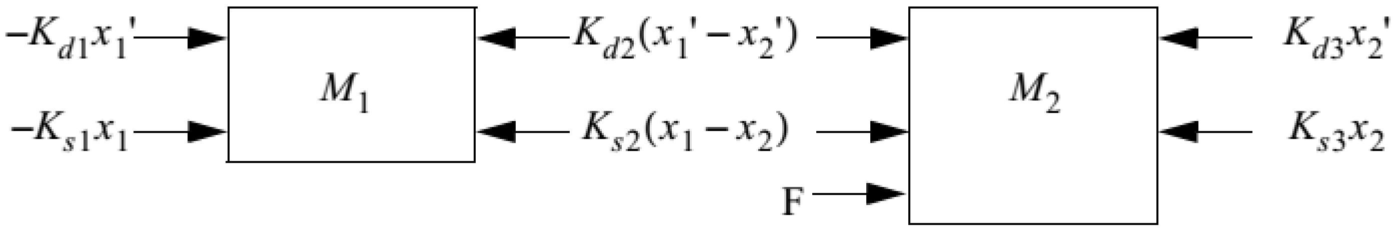


Modeling of Mechanical system

Mathematical Models for the Schematic



- Free Body Diagram FBD

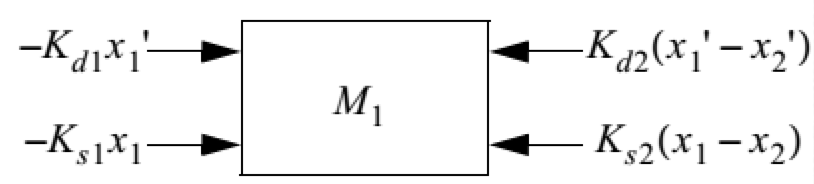


Modeling of Mechanical system

Write equation of motion: Two degree of freedom

Assume X1 >X2 positive direction of motion $\xrightarrow{+}$

- For mass(1)



$$\sum F = -K_{d1}x_1' - K_{s1}x_1 - K_{d2}(x_1' - x_2') - K_{s2}(x_1 - x_2) = M_1 x_1''$$

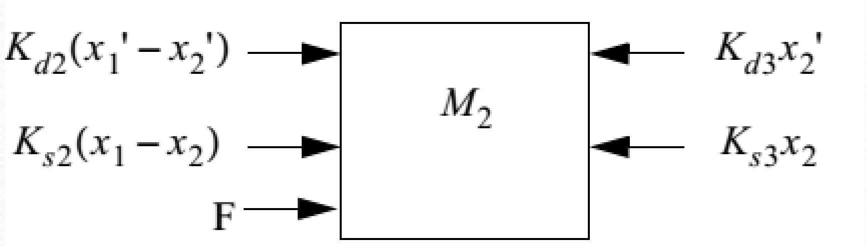
$$x_1''(M_1) + x_1'(K_{d1} + K_{d2}) + x_1(K_{s1} + K_{s2}) + x_2'(-K_{d2}) + x_2(-K_{s2}) = 0$$

Modeling of Mechanical system

Write equation of motion: Two degree of freedom

Assume X1 >X2 positive direction of motion $\xrightarrow{+}$

- For mass(2)



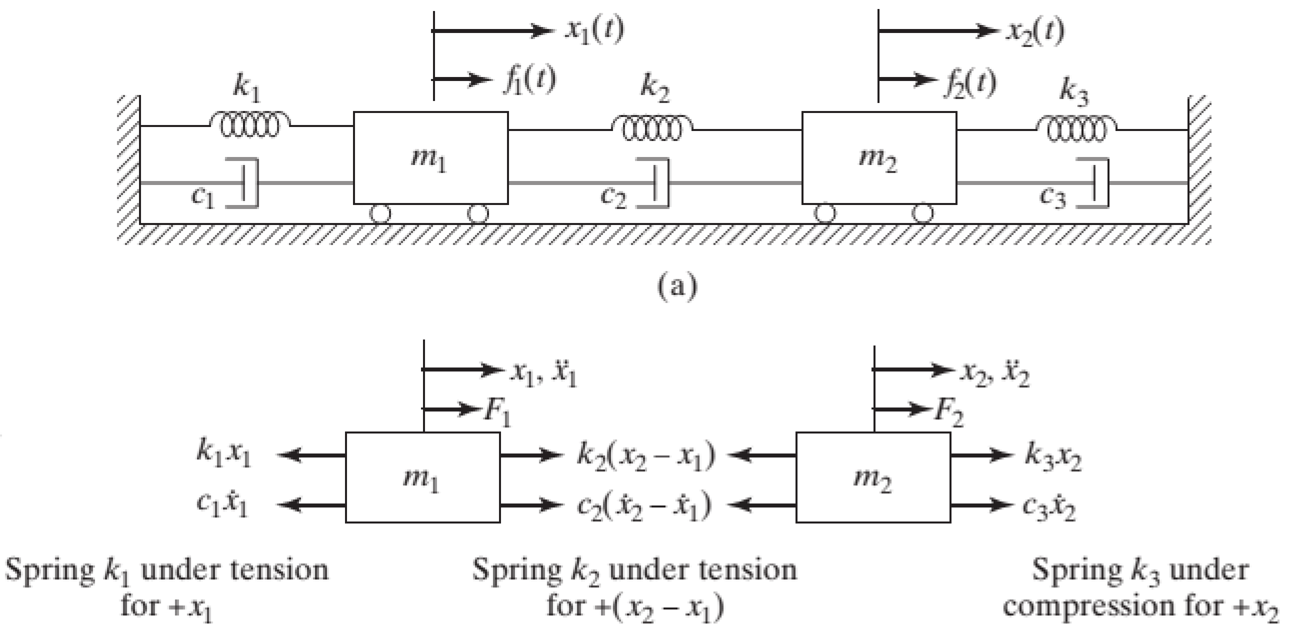
$$\sum F = K_{d2}(x_1' - x_2') + K_{s2}(x_1 - x_2) + F - K_{d3}x_2' - K_{s3}x_2 = M_2 x_2''$$

$$x_2''(M_2) + x_2'(K_{d2} + K_{d3}) + x_2(K_{s2} + K_{s3}) + x_1'(-K_{d2}) + x_1(-K_{s2}) = F$$

Two-Degree-of Freedom Systems

Equations of motion:

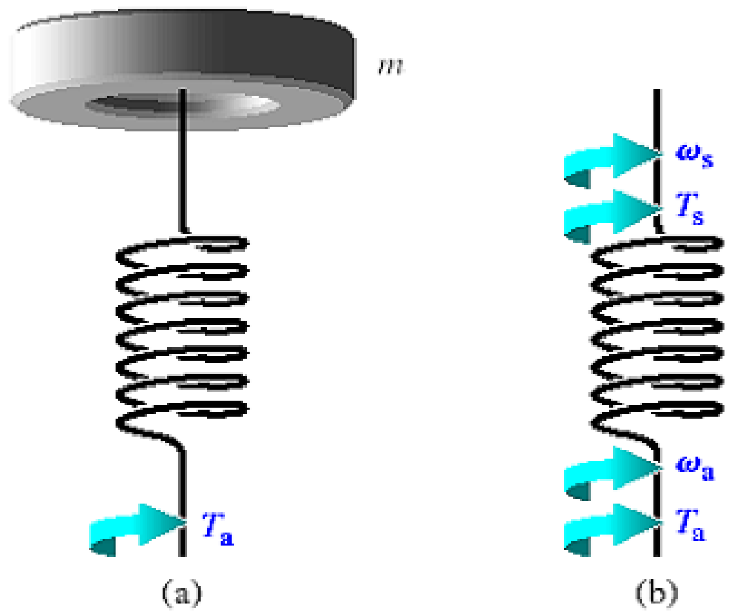
$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$



$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

Modeling of Mechanical System



(a) Torsional spring-mass system.
(b) Spring element.

$T_a(t) - T_s(t) = 0$

$T_a(t) = T_s(t)$

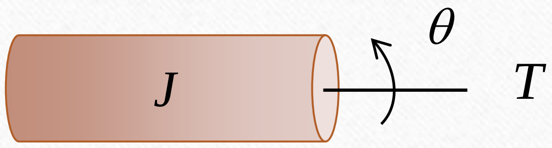
$\omega(t) = \omega_s(t) - \omega_a(t)$

$T_a(t)$ = through - variable

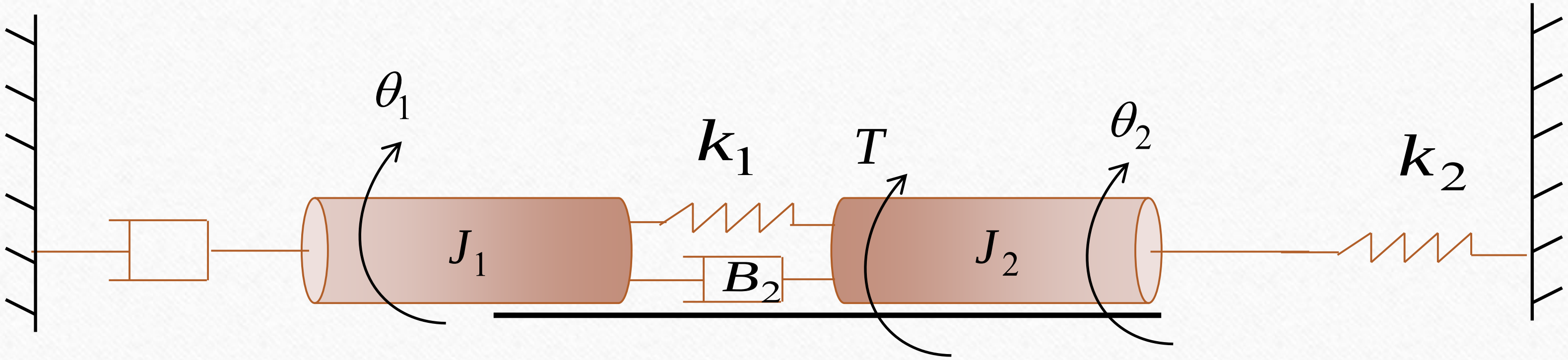
angular rate difference = across-variable

Example

Moment of Inertia



$$T = J\ddot{\theta}$$



Two-Degree-of Freedom Systems

Equations of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

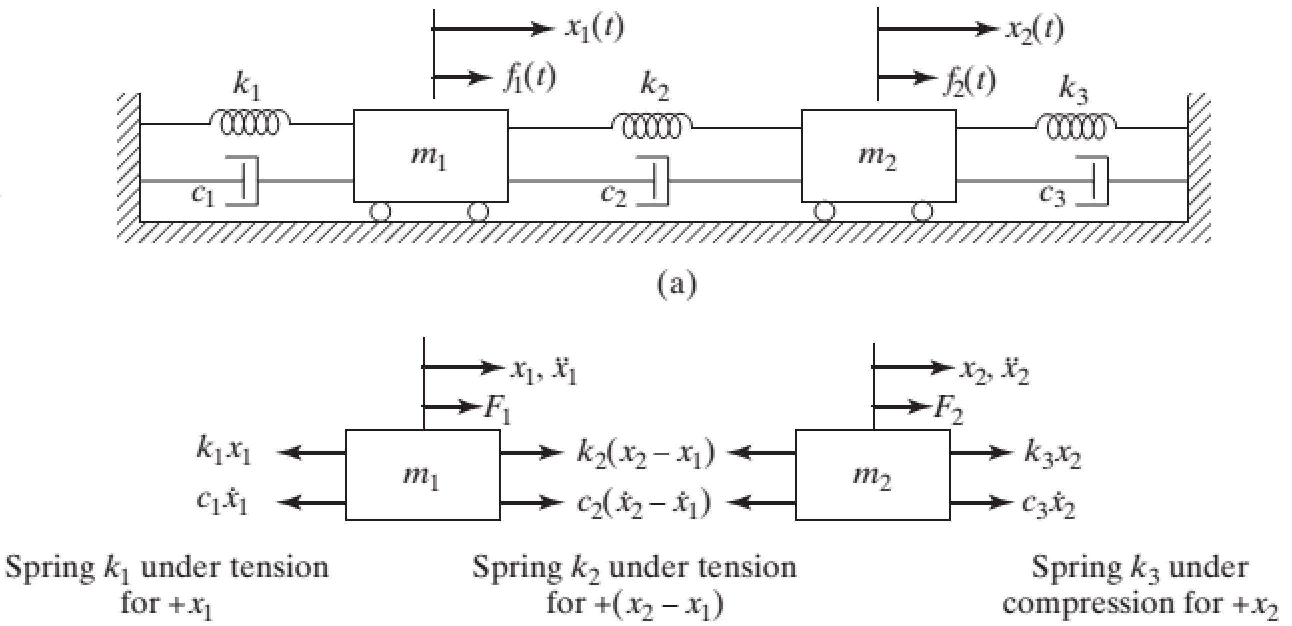
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

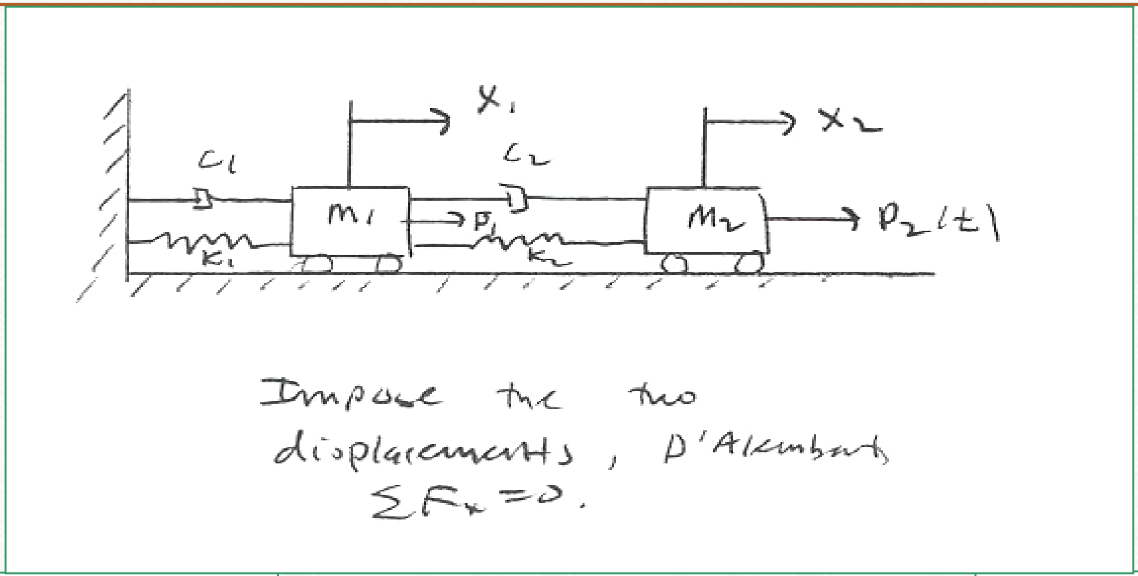
$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

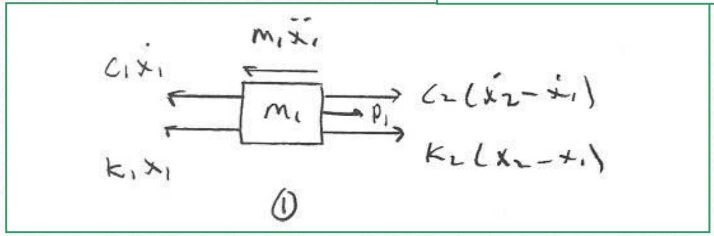
$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

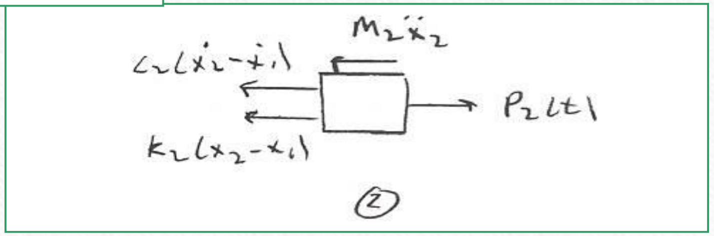


Two-DOF model problem

Matrix form of governing equation:

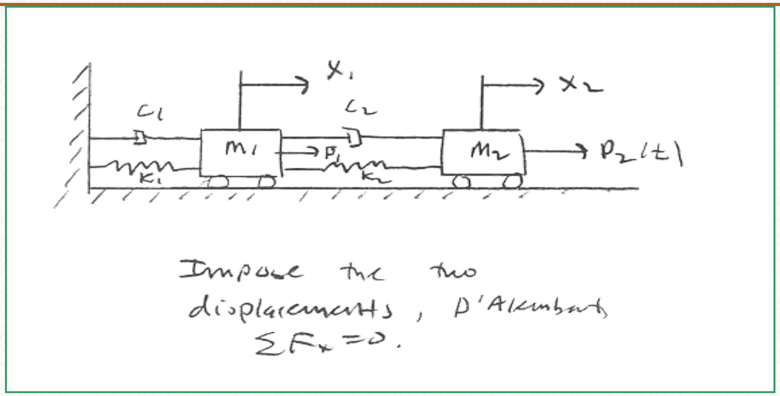


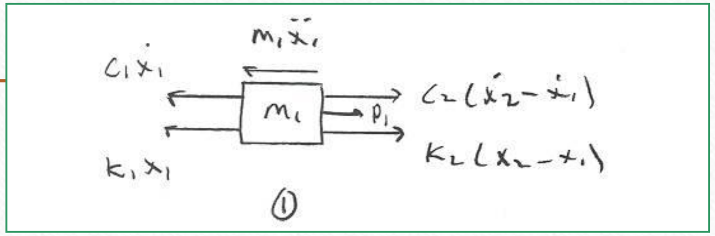


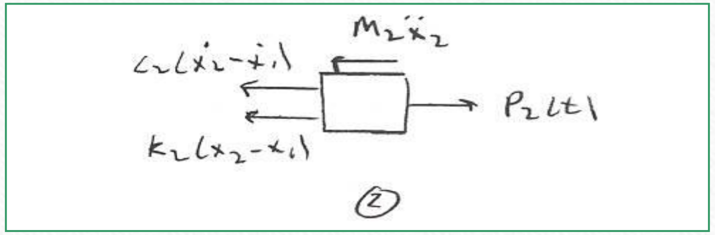


Two-DOF model problem

<u>Matrix form of governing equation:</u>







$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

where: [M] = mass matrix; [C] = damping matrix;
[K] = stiffness matrix; {P} = force vector

<u>Note:</u> Matrices have positive diagonals and are symmetric.

Undamped free vibrations

Zero damping matrix [C] and force vector {P}

- Equation of motion (free-undamped): $M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$
- Assumed general solutions: $$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \cos(\omega t - \phi)$$
- Differentiating twice with respect to time: (Acceleration) $$\{\ddot{x}\} = -\omega^2 \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \cos(\omega t - \phi)$$
- Substitute in equation of motion: (Characteristic equation) $$(K - \omega^2 M)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Undamped free vibrations

Zero damping matrix [C] and force vector {P}

- Characteristic equation:

$$\begin{bmatrix} (k_1+k_2-m_1\omega^2) & -k_2 \\ -k_2 & (k_2-m_2\omega^2) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Characteristic polynomial (for det[ ]=0):

$$\omega^4 - \left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2} \right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

- Eigenvalues (characteristic values):

$$\lambda_{\frac{1}{2}} = \omega_{\frac{1}{2}}^2 = \frac{1}{2}\left\{ \left[\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2} \right] \pm \left[\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{4k_1 k_2}{m_1 m_2} \right]^{1/2} \right\}$$

Undamped free vibrations

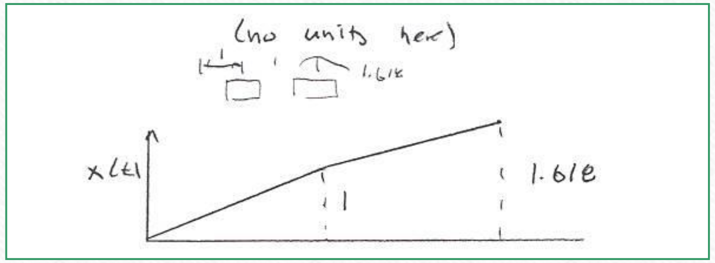
Special case when $k_1=k_2=k$ and $m_1=m_2=m$

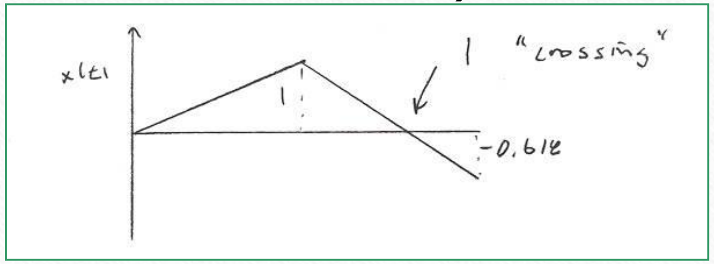
- Eigenvalues and frequencies: $\lambda_{\substack{1\\2}} = \begin{Bmatrix} \omega_1^2 \\ \omega_1^2 \end{Bmatrix} = \begin{Bmatrix} 0.3819 \\ 2.618 \end{Bmatrix} \dfrac{k}{m}$

$$\omega_1 = 0.618\sqrt{\frac{k}{m}} = \text{fundamental frequency}$$

$$T = \frac{2\pi}{\omega} = \text{fundamental period}$$

- Two mode shapes (relative participation of each mass in the motion):





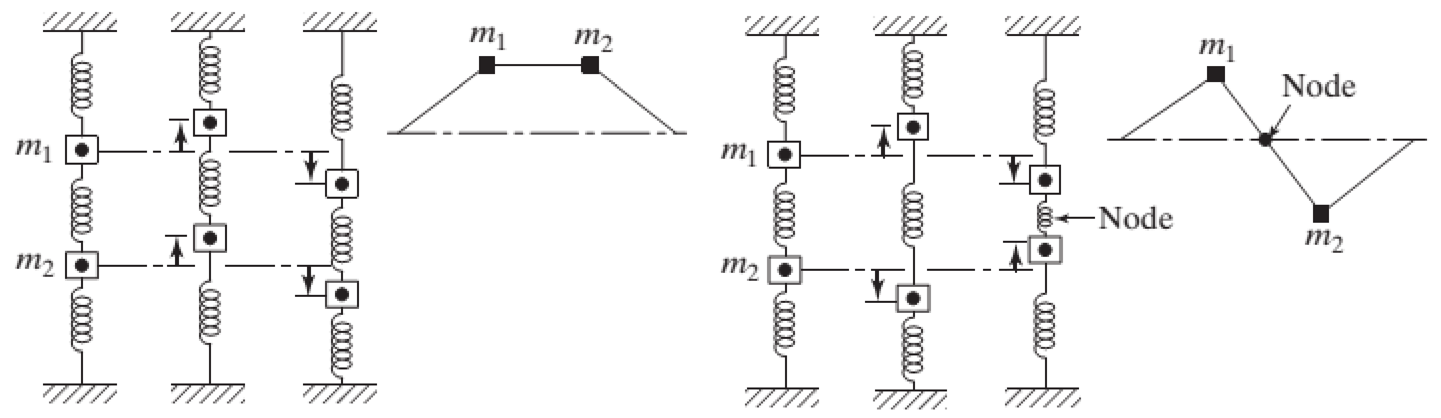
1st mode shape $\dfrac{A_2}{A_1} = \dfrac{2k - m\omega^2}{k} = \dfrac{1.618}{1}$

2nd mode shape $\dfrac{A_2}{A_1} = \dfrac{k}{k - m\omega^2} = \dfrac{-0.618}{1}$

- The two eigenvectors are orthogonal: Eigenvector (1) = $\begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}$ Eigenvector (2) = $\begin{Bmatrix} A_1^{(2)} \\ A_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$

Undamped free vibrations

- Two mode shapes (relative participation of each mass in the motion):



(a) First mode

(b) Second mode

Undamped free vibrations (UFV)

Single-DOF: $x(t) = C\cos(\omega_n t + \phi)$

For two-DOF: $\{x\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \end{Bmatrix} \cos(\omega_1 t + \phi_1) + C_2 \begin{Bmatrix} A_1^{(2)} \\ A_2^{(2)} \end{Bmatrix} \cos(\omega_2 t + \phi_2)$

For any set of initial conditions:

- We know $\{A\}^{(1)}$ and $\{A\}^{(2)}$, ω_1 and ω_2
- Must find C_1, C_2, ϕ_1, and ϕ_2 – Need 4 I.C.'s

Multi-DOF model equation

Multi-DOF systems are so similar to two-DOF.

Model equation: $[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{Q\}$

We derive using:

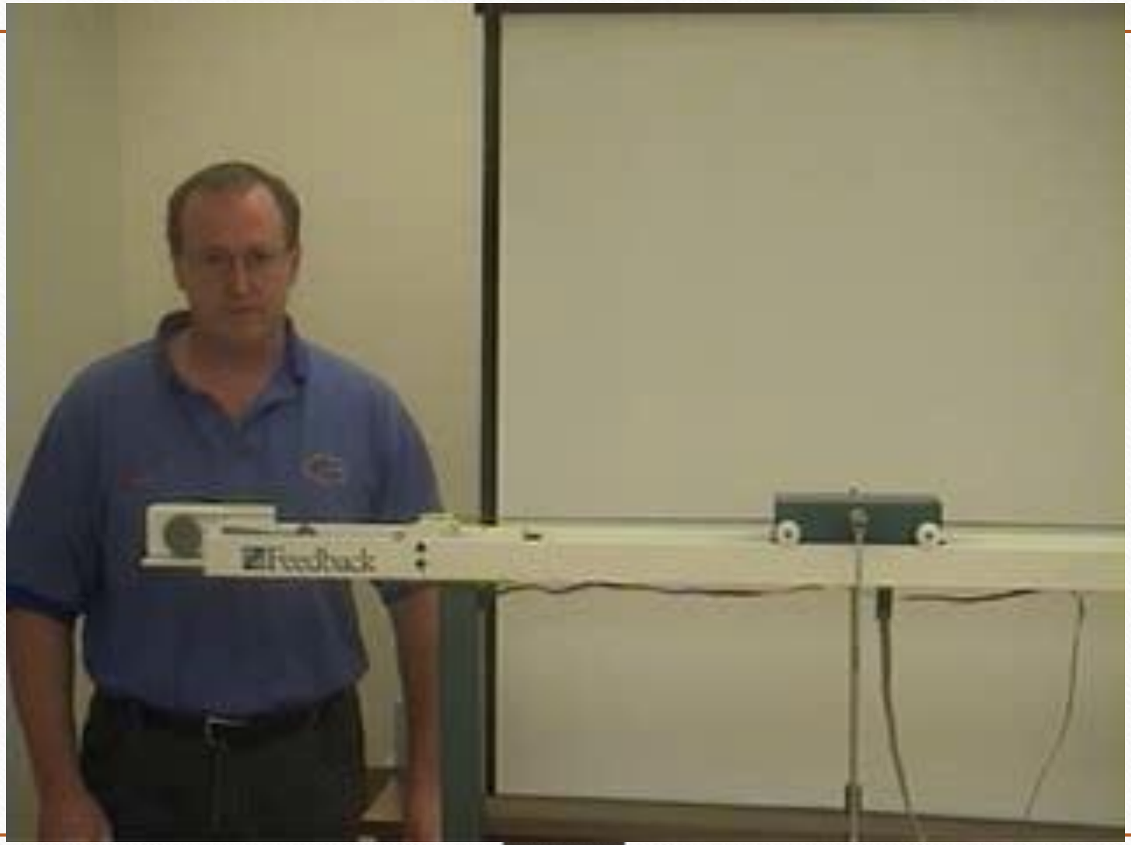
1) Vector mechanics (Newton or D' Alembert)
2) Hamilton's principles
3) Lagrange's equations

Notes on matrices:

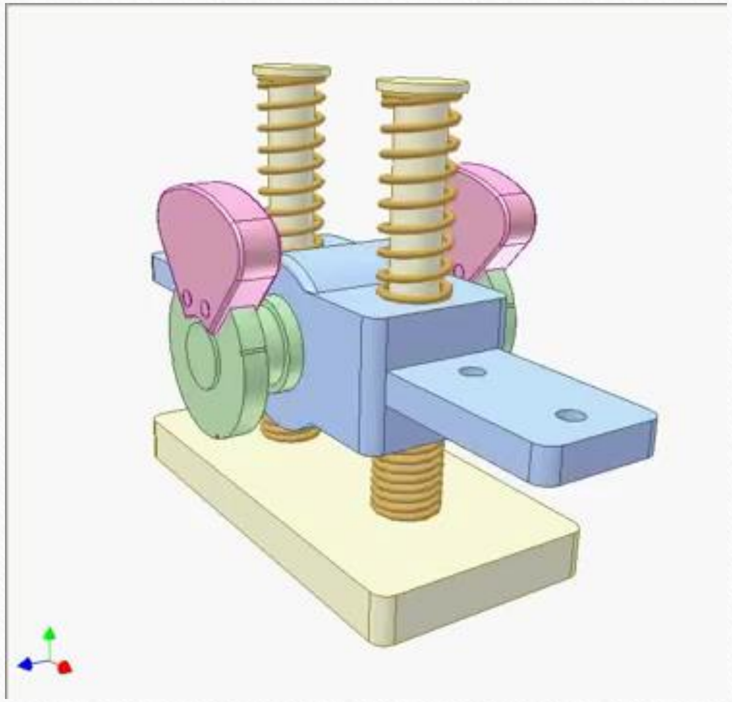
- They are square and symmetric.
- Kinetic energy : $T=\frac{1}{2}\{\dot{x}\}^T[M]\{\dot{x}\}$

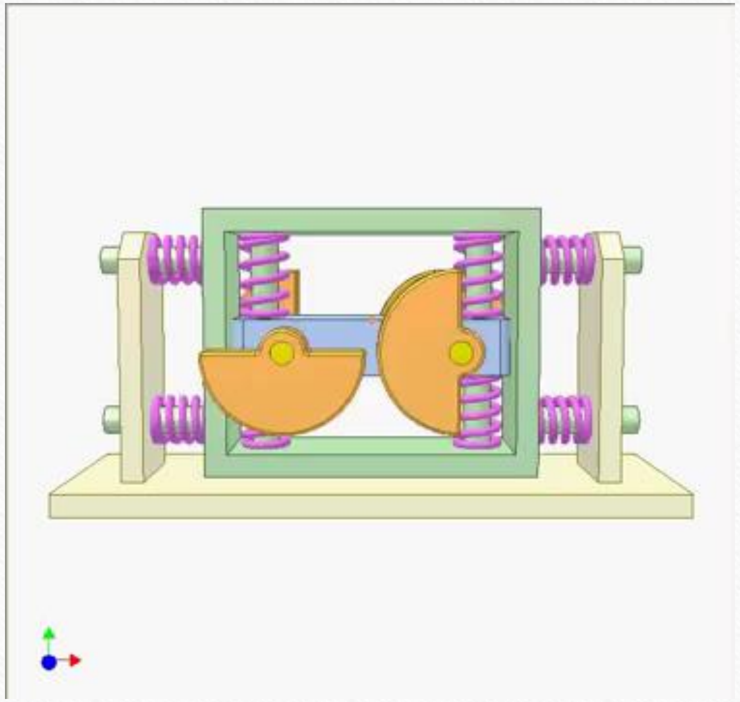
  Strain energy in spring: $U=\frac{1}{2}\{x\}^T[K]\{x\}$
- [M] is positive definite (since T is always positive)
- [K] is positive semi-definite:
  - all positive eigenvalues, except for some potentially 0-eigenvalues which occur during a rigid-body motion.
  - If restrained/tied down $\Rightarrow$ positive-definite. All positive.

Inverted Pendulum

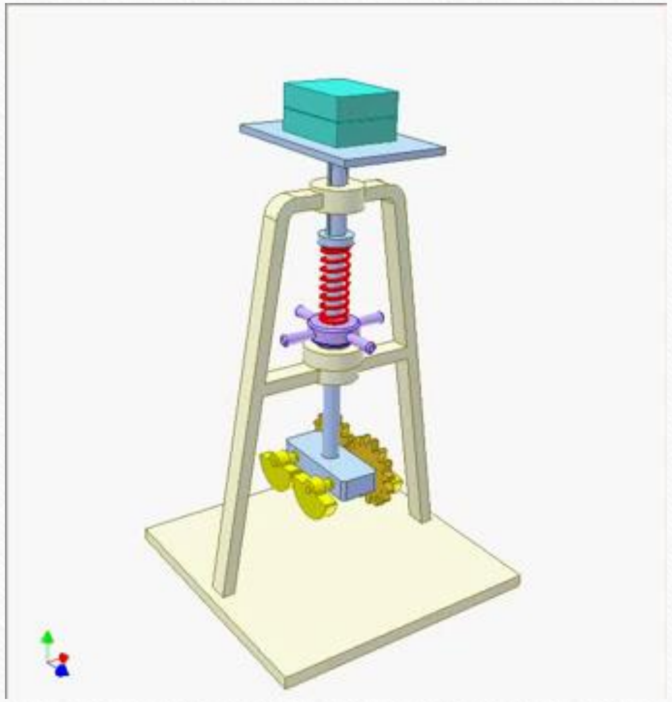


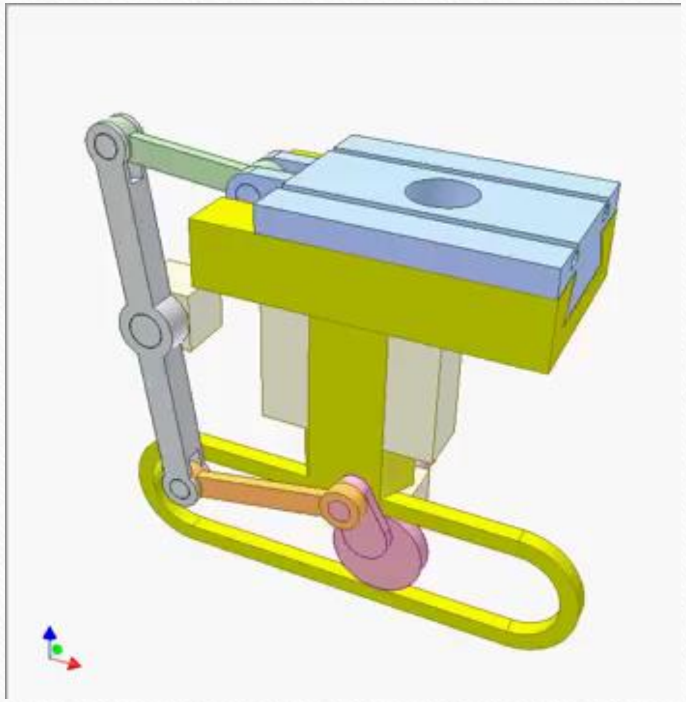
Projects



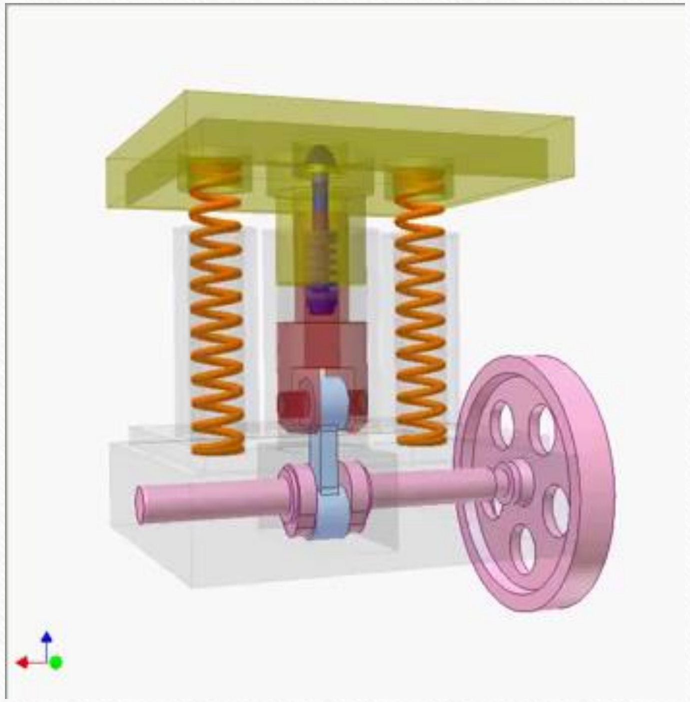


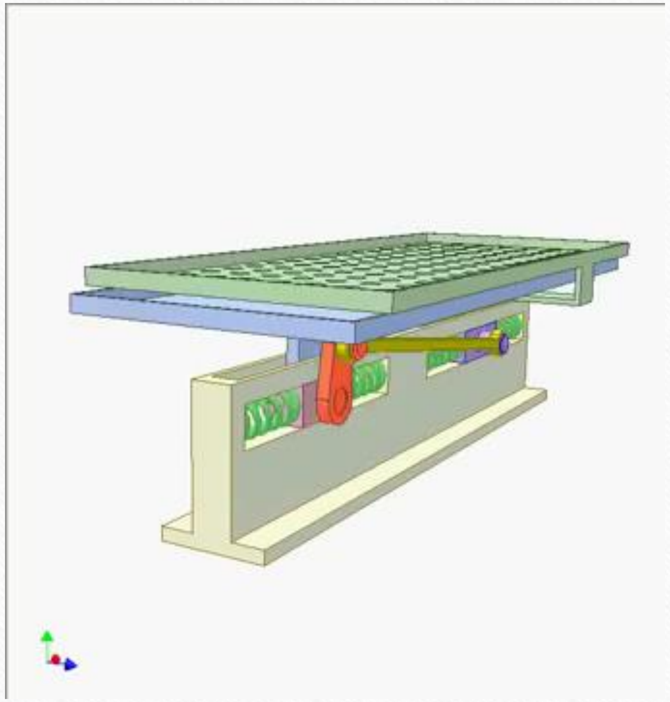
Projects



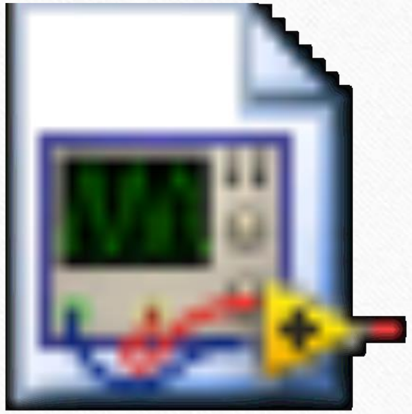


Projects





Program



Express Vibration Lab.vi